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Impulsive Synchronization of Lü Chaotic System Based on Small Impulsive Signal

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Abstract In this Letter the issue of impulsive synchronization of the Lü chaotic system is developed. We propose an impulsive synchronization scheme of the Lü chaotic system including chaotic systems. Some new and sufficient conditions on varying impulsive distance are established in order to guarantee the synchronizability of the systems using the synchronization method. In particular, some simple conditions are derived in synchronizing the systems by equal impulsive distances. The proposed impulsive synchronization scheme is applied to the Lü chaotic system and the numerical example demonstrates the effectiveness of the method. The boundaries of the stable regions are also estimated.

Keywords Chaos · Impulsive synchronization · Lü chaotic system

1 Introduction

The phenomenon on chaos synchronization was first revealed by Pecora and Carroll [1]. Since then, a wide variety of approaches have been proposed for synchronization of chaotic systems, which include drive-response control [1], coupling control [2, 3], feedback control [4–8], adaptive control [9, 10], fuzzy control [11], observer-based control [12], manifold-based method [13, 14], and impulsive control [15–19]. Feedback controller is easy to design, but the controller cannot adapt to the cases with unknown parameters. Adaptive synchronization is useful for synchronization of two parameter-mismatched systems. On the contrary, impulsive controller seems to have a simple structure, and the controller is discontinuous which can be useful for digital communication systems. The research works on impulsive synchronization of [15–17] are based on the theory of comparison systems. But it

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is difficult to estimate the interval of the impulsive control for some systems using this theory. The impulsive synchronization of Chua's oscillator and a hyperchaotic circuit has been studied in [18]. Their experimental results show that the accuracy of impulsively controlled synchronization depends on both the period and the width of the impulse. Furthermore, the robustness of impulsive synchronization to additive noise is also experimentally studied in [18]. Itoh et al. have also given a sufficient condition for impulsive synchronization of continuous systems under the assumption that the synchronization errors are sufficiently small, but this result does not hold for chaotic systems with strong nonlinearities [19].

The impulsive synchronization method is also suitable to deal with systems which cannot endure continuous disturbance. Using this method the slave system receives the information from the master system only in discrete times and the amount of conveyed information is, therefore, decreased. However, this is suitable in practice because of reduced control cost. In this Letter synchronization of the Lü chaotic system via small impulsive signal is investigated. Here, some new and less conservative criteria are proposed to synchronize the systems with varying impulse distances. Particularly, a simple and sufficient condition is derived to achieve the synchronization based on equal impulse distance. The boundaries of the stable regions are also determined. A numerical example is presented to illustrate the feasibility and effectiveness of the method.

2 Theory of Impulsive Synchronization

In the impulsive synchronization, the master system is described by the following relation:

$$\dot{x} = f(t, x) \tag{1}$$

$f : R_+ \times R^n \to R^n$ is a continuous function with respect to its arguments and $x \in R^n$ represents the state variables. The slave system is characterized by

$$\begin{aligned} &\dot{y} = f(t, y), \quad t \neq t_i, \\ &\Delta y = y(t_i^+) - y(t_i^-) = y(t_i^+) - y(t_i) = B_i e, \quad t = t_i, \\ &y(t_0^+) = y_0, \quad i = 1, 2, 3, \ldots. \end{aligned} \tag{2}$$

f is the same function as above, $y \in R^n$ is left continuous at $t = t_i$, B_i are $n \times n$ matrices, and $e = [e_1, e_2, \ldots, e_n]^{\mathrm{T}} = [y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n]^{\mathrm{T}}$. Define a discrete instant set $\{t_i\}$ that satisfies $t_1 < t_2 < \cdots < t_i < t_{i+1} < \cdots$, $t_i \to \infty$ as $i \to \infty$. t_i is the discrete instant that master signal is transmitted to the slave system. The states of the slave system are changed at these instants in accordance to the synchronization errors. Subtracting (2) from (1), one gets the following results for synchronization error dynamics. Since states of the master system are continuous in time. Δx will be zero at time instants t_i

$$\begin{aligned} &\dot{e} = f(t, y) - f(t, x), \quad t \neq t_i, \\ &\Delta e = B_i e, \quad t = t_i. \end{aligned} \tag{3}$$

The goal is to find some conditions on the control gains, B_i and the impulsive distances $\tau_{i+1} = t_{i+1} - t_i < \infty$ $(i = 1, 2, 3, \ldots)$ such that the slave system (2) is synchronized asymptotically with the master system (1) for any initial condition.

3 The Impulsive Synchronization of the Lü Chaotic System

Here we investigate the impulsive synchronization of the Lü chaotic system [20]. The system is described as follows:

$$\begin{aligned}\dot{x}_1 &= a(x_2 - x_1),\\ \dot{x}_2 &= -x_1x_3 + cx_2,\\ \dot{x}_3 &= x_1x_2 - bx_3.\end{aligned} \tag{4}$$

The Lü chaotic system shows chaotic behavior as shown in [14] when $a = 36$, $b = 3$, and $12.7 < c < 17.0$, $18.0 < c < 22.0$, $23.0 < c < 28.5$, $28.6 < c < 29.0$, $29.2334 < c < 29.345$.

First we decompose the system dynamics to its linear and nonlinear parts. Thus (4) is rewritten as

$$\dot{x} = Ax + \phi(x), \tag{5}$$

where $\phi(x)$ represents the nonlinear part of the dynamics,

$$A = \begin{bmatrix} -a & a & 0\\ 0 & c & 0\\ 0 & 0 & -b \end{bmatrix}, \qquad \phi(x) = \begin{bmatrix} 0\\ -x_1x_3\\ x_1x_2 \end{bmatrix}.$$

Therefore, the error dynamics in (3) can be written as

$$\begin{aligned}\dot{e} &= Ae + \psi(x,y), \quad t \neq t_i,\\ \Delta e &= B_i e, \quad t = t_i,\end{aligned} \tag{6}$$

in which

$$\psi(x,y) = \phi(y) - \phi(x) = \begin{bmatrix} 0\\ -y_1y_3 + x_1x_3\\ y_1y_2 - x_1x_2 \end{bmatrix} \tag{7}$$

and t_i are the instants that the impulsive controls are implemented.

Remark 1 To reach a stable equilibrium of (6) Δe should be zero, i.e., $e(t_i^+) = e(t_i^-)$. Since B_i is full rank in general then to reach $e(t_i^+) = e(t_i^-)$ or $\Delta e = 0$, origin $e = 0$ is the unique choice. In other words, the origin is the unique equilibrium of system (6).

Regardless of their initial conditions, chaotic systems have bounded states so that one can find a positive number M such that $|x_i(t)| \leq M$ and $|y_i(t)| \leq M$ for any initial conditions. This fact is used in proof of the following theorem.

Theorem *Let β_i and $\lambda(c)$ be the largest eigenvalues of $(I + B_i)^{\mathrm{T}}(I + B_i)$, $i = 1, 2, 3, \ldots$, and $0.5(A + A^{\mathrm{T}})$, respectively. If there exists a constant $\xi > 1$ such that*

$$\ln(\xi\beta_i) + 2(\lambda(c) + M)\tau_i \leq 0, \quad i = 1, 2, 3, \ldots, \tag{8}$$

then the slave system (2) will be globally asymptotically synchronous with the master system (1).

Proof Let the candidate Lyapunov function be in the form of

$$V(e)=0.5e^{\mathrm{T}}e. \tag{9}$$

The time derivative along the trajectory (7) is

$$\begin{aligned}
\dot V(e)&=0.5(\dot e^{\mathrm{T}}e+e^{\mathrm{T}}\dot e)\\
&=0.5(Ae+\psi(x,y))^{\mathrm{T}}e+0.5e^{\mathrm{T}}(Ae+\psi(x,y))\\
&=0.5e^{\mathrm{T}}(A^{\mathrm{T}}+A)e-e_1e_2y_3+e_1e_3y_2\\
&\le 2\lambda(c)V(e(t))+M(|e_1e_2|+|e_1e_3|)\\
&\le 2(\lambda(c)+M)V(e(t)),
\end{aligned} \tag{10}$$

$t\in(t_{i-1},t_i]$ for $i=1,2,3,\ldots$.

This implies that $V(e(t))\le V(e(t_{i-1}^+))e^{2(\lambda(c)+M)(t-t_{i-1})}$, $t\in(t_{i-1},t_i]$ for $i=1,2,\ldots$. Now from (6)

$$\begin{aligned}
V(e(t_i^+))&=0.5[(I+B_i)e(t_i)]^{\mathrm{T}}(I+B_i)e(t_i)\\
&=0.5e^{\mathrm{T}}(t_i)[(I+B_i)^{\mathrm{T}}(I+B_i)]e(t_i)\\
&\le 0.5\beta_ie^{\mathrm{T}}(t_i)e(t_i)=\beta_iV(e(t_i)).
\end{aligned} \tag{11}$$

When $i=1$ in inequality (11), then for any $t\in(t_0,t_1]$, $V(e(t))\le V(e(t_0^+))e^{2(\lambda(c)+M)(t-t_0)}$.

This leads to $V(e(t_1))\le V(e(t_0^+))e^{2(\lambda(c)+M)(t_1-t_0)}$. Also from (11) we have

$$V(e(t_1^+))\le\beta_1V(e(t_1))\le\beta_1V(e(t_0^+))e^{2(\lambda(c)+M)(t_1-t_0)}. \tag{12}$$

In the same way for $t\in(t_1,t_2]$ we have

$$V(e(t))\le V(e(t_1^+))e^{2(\lambda(c)+M)(t-t_1)}\le\beta_1V(e(t_0^+))e^{2(\lambda(c)+M)(t-t_0)}. \tag{13}$$

In general for any $t\in(t_i,t_{i+1}]$ one finds that

$$V(e(t))\le\beta_1\beta_2\cdots\beta_iV(e(t_0^+))e^{2(\lambda(c)+M)(t-t_0)}. \tag{14}$$

From the assumption given in the theorem we have

$$\xi\beta_ie^{2(\lambda(c)+M)\tau_i}\le 1,\quad i=1,2,\ldots. \tag{15}$$

Thus for $t\in(t_i,t_{i+1}]$, $i=1,2,\ldots$, we have

$$\begin{aligned}
V(e(t))&\le\beta_1\beta_2\cdots\beta_iV(e(t_0^+))e^{2(\lambda(c)+M)(t-t_0)}\\
&=V(e(t_0^+))[\beta_1e^{2(\lambda(c)+M)\tau_1}]\cdots[\beta_ie^{2(\lambda(c)+M)\tau_i}]e^{2(\lambda(c)+M)(t-t_i)}\\
&\le V(e(t_0^+))\frac{1}{\xi^i}e^{2(\lambda(c)+M)(t-t_i)}.
\end{aligned} \tag{16}$$

This implies that the origin in system (6) is globally asymptotically stable or the slave system is synchronized with the master system asymptotically for any initial conditions. By this we conclude proof of the theorem. □

To be convenient the gain matrices B_i and the impulsive distances τ_i can be chosen constant. Thus we have the following corollary.

Corollary *Suppose $\tau_i = \tau > 0$ and gain matrices $B_i = B$ $(i = 1, 2, \ldots)$. If there exists a constant $\xi > 1$ such that*

$$\ln(\xi\beta) + 2(\lambda(c) + M)\tau \leq 0 \tag{17}$$

then the slave system (2) is globally asymptotically synchronous with the master system (1).

4 Numerical Simulations

In order to demonstrate and verify the performance of the proposed method, two numerical simulations are presented in this section. The Lü chaotic system is given in (4) where a, b and c are the real constants. Typical phase portraits of this system are shown as Fig. 1. For simulation purpose we choose $a = 36$, $b = 3$, $c = 13.0$, it is a forced dissipative system with bounded states ($M \leq 25$) as $t \to \infty$, then

$$0.5(A^{\mathrm{T}} + A) = \begin{bmatrix} -36 & 18 & 0 \\ 18 & 13 & 0 \\ 0 & 0 & -3 \end{bmatrix}. \tag{18}$$

Thus, $\lambda(13.0) = 18.9015$. And suppose B_i be a constant matrix

$$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix},$$

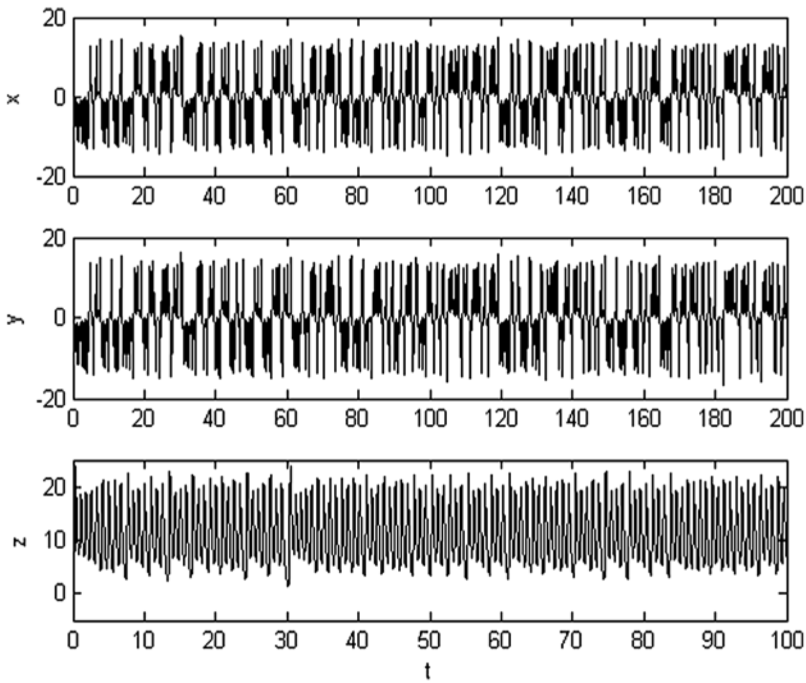

Fig. 1 Chaotic behavior obtained by integrating numerically the system (4). The parameter values are $a = 36$, $b = 3$, $c = 13$, initial conditions are 0.5, 1, 0.5

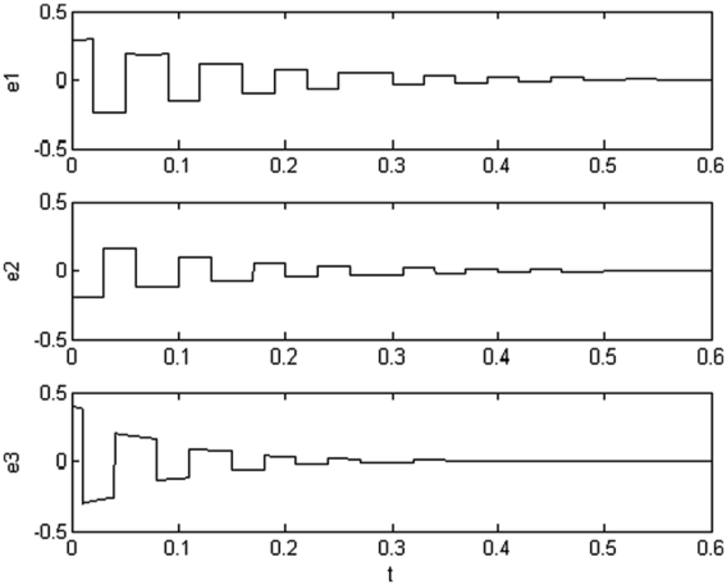

Fig. 2 Time response of the synchronization error system with $k=-1.2$, $\xi=3$, and $\tau=0.01$

it is evident that $\beta=(1+k)^2$. From (17), the estimation of bounds of stable regions are given by

$$0 \le \tau \le -\frac{\ln \xi + \ln(k+1)^2}{87.8030}. \tag{19}$$

Figures 2 and 3 show the time response curves of synchronization error systems. For example if $k=-1.2$, $\xi=3$, then $0 \le \tau \le 0.0241$. The numerical simulation results for this case are shown in Fig. 2. In the second simulation we choose the gain matrix $B_i = B$ a diagonal matrix as follows:

$$B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1.2 & 0 \\ 0 & 0 & -0.6 \end{bmatrix}. \tag{20}$$

For this section we have $\beta = 0.16$. For $\xi = 1.16$, and $\tau = 0.01$, thus condition given in (18) is satisfied. Figure 3 shows results of the computer simulation in this case. In simulations, we choose the parameters of system (4) as $a=36$, $b=3$, $c=13.0$. A fourth-order Runge-Kutta method with step size 10^{-5} is used. The initial conditions for driving and driven systems are given by $x(0)=(0.1,0.1,0.2)^{\mathrm{T}}$ and $\tilde{x}(0)=(-0.3,-0.5,0.6)^{\mathrm{T}}$, respectively.

Remark 2 We can see that the stable regions in this paper are simpler and bigger than in Theorem 4 in [21].

5 Conclusion

In this paper, we have investigated the issue on the synchronization of Lü systems via an impulsive method. Some simple conditions are obtained in synchronizing the systems by equal

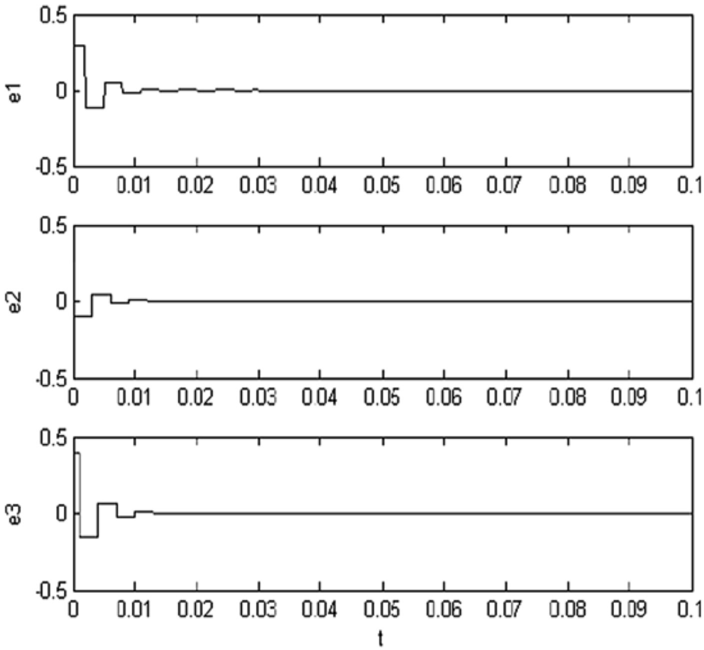

Fig. 3 Time response of the synchronization error system with $\beta = 0.16$, $\xi = 1.16$, and $\tau = 0.01$

impulsive distances to guarantee the impulsive synchronization globally asymptotically synchronous. An illustrative example has shown satisfying synchronization performance.

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References

1. Pecora, L.M., Carroll, T.L.: Phys. Rev. Lett. **64**, 821 (1990)
2. Heagy, J.F., Carroll, T.L., Pecora, L.M.: Phys. Rev. E **52**, 3240 (1995)
3. Heagy, J.F., Carroll, T.L., Pecora, L.M.: Phys. Rev. Lett. **73**, 3528 (1994)
4. Peng, Z.W., Zhong, T.X.: Chin. Phys. **4**, 244 (2000)
5. Zhang, J.S., Wan, J.H., Xiao, X.C.: Chin. Phys. **2**, 97 (2001)
6. Li, L.X., Peng, H.P., Lu, H.B., Guan, X.P.: Chin. Phys. **9**, 796 (2001)
7. Wei, R., Wang, X.Y.: Physics **8**, 1210 (2004)
8. Lu, J.G.: Chin. Phys. **7**, 1342 (2005)
9. Li, Z., Han, C.Z.: Chin. Phys. **6**, 494 (2001)
10. Chen, S.H., Zhao, L.M., Liu, J.: Chin. Phys. **6**, 543 (2002)
11. Wang, Y.N., Tan, W., Duan, F.: Chin. Phys. **1**, 89 (2006)
12. Sanchez, E.N., Perez, J.P., Ricalde, L.J., Chen, G.: In: Proceedings of the 40th IEEE Conference on Decision and Control, vol. 40, p. 3536 (2001)
13. Fang, J.Q., Hong, Y., Chen, G.: Phys. Rev. E **59**, 2523 (1999)
14. Yu, X.H., Chen, G.R., Xia, Y., et al.: IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. **48**, 930 (2001)
15. Yang, T., Yang, L.B., Yang, C.M.: Physica D **110**, 18 (1997)
16. Yang, T., Chua, L.O.: IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. **44**, 976 (1997)
17. Xie, W., Wen, C., Li, Z.: Phys. Lett. A **257**, 67 (2000)
18. Ioth, M., Yang, T., Chua, L.O.: Int. J. Bifurc. Chaos Appl. **9**, 1393 (1999)

19. Ioth, M., Yang, T., Chua, L.O.: Int. J. Bifurc. Chaos Appl. **11**, 551 (2001)
20. Lü, J.H., Chen, G.R.: Int. J. Bifurc. Chaos Appl. **12**, 659 (2002)
21. Zhang, Y.P., Sun, J.T.: Phys. Lett. A **342**, 256 (2005)